SOGANG-HEP 269/00
hep-th/0003093

Comments on Instantons on Noncommutative $\mathbf{R}^4$

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ABSTRACT

We study $U(1)$ and $U(2)$ instanton solutions on noncommutative $\mathbf{R}^4$ based on the noncommutative version of ADHM equation proposed by Nekrasov and Schwarz. It is shown that the anti-self-dual gauge fields on self-dual noncommutative $\mathbf{R}^4$ correctly give integer instanton numbers for all cases we consider. We also show that the completeness relation in the ADHM construction is generally satisfied even for noncommutative spaces.

February 1, 2008

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1 Introduction

Recently it has been known [1, 2, 3] that quantum field theory on noncommutative space can arise naturally as a decoupled limit of open string dynamics on D-branes in the background of Neveu-Schwarz two-form field B_{NS}. In particular, it was shown in [1, 2] that noncommutative geometry can be successfully applied to the compactification of M(atrix) theory [4, 5] in a certain background and the low energy effective theory for D-branes in the B_{NS} field background, which are specifically described by a gauge theory on noncommutative space [3].

In their paper [6], Nekrasov and Schwarz showed that instanton solutions in noncommutative Yang-Mills theory can be obtained by Atiyah-Drinfeld-Hitchin-Manin (ADHM) equation [7] defined on noncommutative $\mathbf{R}^4$ which is equivalent to adding a Fayet-Iliopoulos term to the usual ADHM equation. The remarkable fact is that the deformation of the ADHM equation has an effect removing the singularity of the instanton moduli space [6, 8, 9, 10, 11, 12].

In this report, we study $U(1)$ and $U(2)$ instanton solutions on noncommutative $\mathbf{R^4}$ based on the noncommutative version of ADHM equation proposed by Nekrasov and Schwarz. Here it is shown that the anti-self-dual gauge fields on self-dual noncommutative $\mathbf{R}^4$ correctly give integer instanton numbers for all cases we consider. After this paper, in [13], a method to obtain (anti-)self-dual solutions with integer instanton number was proposed using 't Hooft ansatz although the resulting field strength is not Hermitian.

The paper is organized as follows. In next section we review the ADHM construction of noncommutative instantons on self-dual noncommutative $\mathbf{R}^4$. In section 3 we explicitly calculate the anti-self-dual field strengths for simple cases, $k = 1$ and $k = 2$ for $U(1)$ and $k = 1$ for $U(2)$. We show that these solutions correctly give integer instanton numbers. In section 4 we discuss the completeness relation as an ADHM condition. It is also shown that the completeness relation in the ADHM construction is generally satisfied even for noncommutative spaces. In section 5 we discuss the results obtained and address some issues.

Recently, in their paper [14], Chu, et al. pointed out that the topological charge of noncommutative instantons by ADHM construction we considered in this paper is correctly an integer. According to their pointing out, we have redone the numerical calculation on the instanton number using Maple. Now we have obtained integer instanton numbers for all solutions previously found in this paper. It turned out that the previous puzzling result had been caused by an error of our numerical calculation.

2 Noncommutative Version of ADHM Equation

Noncommutative $\mathbf{R}^4$ is described by algebra generated by x^μ obeying the commutation relation:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a non-degenerate matrix of real and constant numbers. Since we are interested in noncommutative instanton backgrounds and the instanton moduli space only depends on the self-dual part $\theta^+ = 1/2(\theta + *\theta)$ [3, 6], we restrict ourselves to the case where $\theta^{\mu\nu}$ is self-dual and set

$$\theta^{12} = \theta^{34} = \frac{\zeta}{4}.$$

Then the algebra in (1), denoted as $\mathcal{A}_\zeta$, depends only on one parameter ζ where we choose $\zeta > 0$. We consider an algebra $\mathcal{A}_\zeta$ consisted of smooth operators $\mathcal{O}$. The commutation relations (1) have an automorphism of the form $x^\mu \mapsto x^\mu + c^\mu$, where c^μ is a commuting real number, and we denote the Lie algebra of this group by $\underline{\mathbf{g}}$. Since the derivative operator ∂_μ can be understood as the action of $\underline{\mathbf{g}}$ on $\mathcal{A}_\zeta$ by translation, the generators of $\underline{\mathbf{g}}$ can be defined by unitary operators $U_c = e^{c^\mu \partial_\mu}$ where $\partial_\mu = -iB_{\mu\nu}x^\nu$ with $B_{\mu\nu}$, an inverse matrix of $\theta^{\mu\nu}$. Then the derivative for a smooth operator $\mathcal{O}$ is defined by $\partial_\mu \mathcal{O} \equiv [\partial_\mu, \mathcal{O}]$. One can check that ∂_μ satisfies following commutation relations

$$[\partial_\mu, x^\nu] = \delta_\mu^\nu, \quad [\partial_\mu, \partial_\nu] = iB_{\mu\nu}.$$

However the action of two derivatives on any operator $\mathcal{O}$ commutes:

$$\partial_\mu \partial_\nu \mathcal{O} - \partial_\nu \partial_\mu \mathcal{O} = [[\partial_\mu, \partial_\nu], \mathcal{O}] = 0.$$

Introduce the generators of noncommutative $\mathbf{C}^2 \approx \mathbf{R}^4$ by

$$z_1 = x^2 + ix^1, \quad z_2 = x^4 + ix^3. \tag{2}$$

Their non-vanishing commutation relations reduce to

$$[\bar{z}_1, z_1] = [\bar{z}_2, z_2] = \frac{\zeta}{2}. \tag{3}$$

The commuatation algebra is that of the annihilation and creation operators for a simple harmonic oscillator (SHO) and so one may use the SHO Hilbert space as a representation of $\mathcal{A}_\zeta$ as adopted in [6, 11]. Therefore let's start with the algebra End $\mathcal{H}$ with finite norm of operators acting on the Fock space $\mathcal{H} = \sum_{(n_1,n_2)\in \mathbf{Z}^2_{\geq 0}} \mathbf{C}|n_1, n_2>$, where $\bar{z}, z$ are represented as

annihilation and creation operators:

$$\sqrt{\frac{2}{\zeta}}\bar{z}_1|n_1,n_2>=\sqrt{n_1}|n_1-1,n_2>, \quad \sqrt{\frac{2}{\zeta}}z_1|n_1,n_2>=\sqrt{n_1+1}|n_1+1,n_2>,$$
$$\sqrt{\frac{2}{\zeta}}\bar{z}_2|n_1,n_2>=\sqrt{n_2}|n_1,n_2-1>, \quad \sqrt{\frac{2}{\zeta}}z_2|n_1,n_2>=\sqrt{n_2+1}|n_1,n_2+1>. \qquad (4)$$

ADHM construction describes a way for finding anti-self-dual configurations of the gauge field in terms of some quadratic matrix equations on $\mathbf{R}^4$ [7]. Recently, N. Nekrasov and A. Schwarz made the ADHM construction to be applied to the noncommutative $\mathbf{R}^4$ [6]. In order to describe k instantons with gauge group $U(N)$, one starts with the following data:
1. A pair of complex hermitian vector spaces $V=\mathbf{C}^k$, $W=\mathbf{C}^N$.
2. The operators $B_1, B_2 \in Hom(V,V)$, $I \in Hom(W,V)$ and $J \in Hom(V,W)$ satisfying the equations

$$\mu_r = [B_1,B_1^\dagger]+[B_2,B_2^\dagger]+II^\dagger - J^\dagger J = \zeta, \qquad (5)$$
$$\mu_c = [B_1,B_2]+IJ=0. \qquad (6)$$

3. Define a Dirac operator $D^\dagger : V\oplus V\oplus W \to V\oplus V$ by

$$D^\dagger = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix} \qquad (7)$$

where

$$\tau_z = \begin{pmatrix} B_2-z_2 & B_1-z_1 & I \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} -B_1+z_1 \\ B_2-z_2 \\ J \end{pmatrix}. \qquad (8)$$

Then the ADHM equations (5) and (6) are equivalent to the set of equations

$$\tau_z\tau_z^\dagger = \sigma_z^\dagger\sigma_z, \quad \tau_z\sigma_z=0. \qquad (9)$$

According to the ADHM construction, we can get the gauge field (instanton solution) by the formula

$$A_\mu = \psi^\dagger\partial_\mu\psi, \qquad (10)$$

where $\psi : W \to V\oplus V\oplus W$ is N zero-modes of $D^\dagger$, i.e.,

$$D^\dagger\psi = 0. \qquad (11)$$

For given ADHM data and the zero mode condition (11), the following completeness relation has to be satisfied to construct an (anti-)self-dual field strength from the gauge field (10)

$$D\frac{1}{D^\dagger D}D^\dagger + \psi\psi^\dagger = 1. \qquad (12)$$

We will show in section 4 that this relation is always satisfied even for noncommutative spaces.

Note that the vector bundle over the noncommutative space $\mathcal{A}_\zeta$ is a finitely generated projective module.[1] It was pointed out in [11, 12] that for a noncommutative space there can be a projective module associated with a projection with non-constant rank (so the corresponding bundle has non-constant dimension) and the instanton module is the kind that is related to a projection operator in $\mathcal{A}_\zeta$ given by

$$p = \psi^\dagger \psi. \tag{13}$$

Just as in the ordinary case, the anti-self-dual field strength F_A can be calculated by the following formula

$$\begin{aligned} F_A &= \psi^\dagger \left(d\tau_z^\dagger \frac{1}{\triangle_z} d\tau_z + d\sigma_z \frac{1}{\triangle_z} d\sigma_z^\dagger \right) \psi \\ &= \psi^\dagger \begin{pmatrix} dz_1 \frac{1}{\triangle_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{\triangle_z} dz_2 & d\bar{z}_2 \frac{1}{\triangle_z} dz_1 - dz_1 \frac{1}{\triangle_z} d\bar{z}_2 & 0 \\ d\bar{z}_1 \frac{1}{\triangle_z} dz_2 - dz_2 \frac{1}{\triangle_z} d\bar{z}_1 & d\bar{z}_1 \frac{1}{\triangle_z} dz_1 + dz_2 \frac{1}{\triangle_z} d\bar{z}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi, \end{aligned} \tag{14}$$

where $\triangle_z = \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z$ has no zero-modes so it is invertible [6, 11]. Note that the completeness relation (12) is used to derive the above field strength F_A which is anti-self-dual.

3 Explicit Calculation of Instanton Charge

In this section, we will perform an explicit calculation on the instanton charge from the solutions obtained by the ADHM construction in the previous section. First, we briefly do it for single $U(1)$ instanton solution obtained by Nekrasov and Schwarz [6] for the purpose of illuminating our calculational method. And then we will do the same calculation for two $U(1)$ instantons and single $U(2)$ instanton solutions.

It is natural to require that the integration on a quantum, i.e. noncommutative, space, which is the trace over its Hilbert space or more precisely Dixmir trace, has to respect a symmetry $x \to x' = x'(x)$, that is,

$$\mathrm{Tr}_{\mathcal{H}}\Big(\mathcal{O}(x')\Big) = \mathrm{Tr}_{\mathcal{H}}\Big(\mathcal{O}(x)\Big).$$

Intuitively, the quantized $\mathbf{R}^4$ in the basis (4) becomes two dimensional integer lattice $\{(n_1, n_2) \in \mathbf{Z}^2_{\geq 0}\}$. Thus one can naturally think that the integration on classical $\mathbf{R}^4$ should be replaced by the sum over the lattice:

$$\int d^4x \mathcal{O}(x) \to \mathrm{Tr}_{\mathcal{H}} \mathcal{O}(x) \equiv \Big(\frac{\zeta\pi}{2}\Big)^2 \sum_{(n_1,n_2)} \langle n_1, n_2 | \mathcal{O}(x) | n_1, n_2 \rangle. \tag{15}$$

[1]The module $\mathcal{E}$ is projective if there exists another module $\mathcal{F}$ such that the direct sum $\mathcal{E} \oplus \mathcal{F}$ is free, i.e., $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}_\zeta^N$ as right $\mathcal{A}_\zeta$-module.

This definition used in [6] to calculate the instanton number of $U(1)$ solution indeed respects the translational symmetry as well as the rotational symmetry since the automorphism $\underline{\mathbf{g}}$ of $\mathbf{R}^4$ such as (3) can be generated by the unitary transformation acting on $\mathcal{H}$. If the system preserves rotational symmetry, for example, single instanton solution, we then expect for the case that the sum with respect to (n_1, n_2) can be reduced to that with respect to $N = n_1 + n_2$ corresponding to a radial variable $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$.

3.1 Single $U(1)$ Instanton

In the ordinary case any regular $U(1)$ instanton solution can not exist. However, in the noncommutative case, there are nontrivial $U(1)$ instantons for every k [6, 11, 10]. Suppose (B_1, B_2, I) is a solution to the equations (5) and (6) where one can show $J = 0$ for $U(1)$ [9]. If we write the element of $V \oplus V \oplus W$ as $\psi = \psi_1 \oplus \psi_2 \oplus \xi$, Eq.(11) is then

$$D^\dagger \psi = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \\ -B_1^\dagger + \bar{z}_1 & B_2^\dagger - \bar{z}_2 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} = 0. \tag{16}$$

For $k = 1$ we can first choose $B_1 = B_2 = 0$ by translation and we get $I = \sqrt{\zeta}$ from the ADHM equation (5). Then Eq.(16) is solved as

$$\psi_1 = \bar{z}_2 \delta^{-1} I \xi, \quad \psi_2 = \bar{z}_1 \delta^{-1} I \xi, \tag{17}$$

where $\delta \equiv x^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2$ and $\xi = (1 + I^\dagger \delta^{-1} I)^{-\frac{1}{2}}$. As emphasized by Furuuchi [11] and Ho [12], note that the operator ψ annihilates $|0,0\rangle$ state, so the projective module is normalized as

$$\psi^\dagger \psi = p \equiv 1 - |0,0\rangle\langle 0,0|. \tag{18}$$

If $\bar{z}_1$, $\bar{z}_2$ in (17) are ordered to the right of $\delta^{-1} I \xi$ according to the commutation rules

$$\bar{z}_\alpha f(\delta) = f(\delta + \zeta/2) \bar{z}_\alpha, \quad z_\alpha f(\delta) = f(\delta - \zeta/2) z_\alpha, \quad (\alpha = 1, 2) \tag{19}$$

for a function $f(z, \bar{z})$, the $|0,0\rangle$ state is projected out from the Fock space and $\delta^{-1} \to (\delta + \zeta/2)^{-1}$. So the ADHM solution (17) is well defined for all states in $\mathcal{H}$. It is crucial to prove the completeness relation (12) to observe that ADHM always arrange their solutions to be singularity-free.

Now we can calculate the connection $A = \psi^\dagger d\psi$ in terms of one variable ξ

$$A = \xi^{-1} \alpha \xi + \xi^{-1} d\xi, \tag{20}$$

where $\alpha = \xi^2 \partial_z \xi^{-2} = -\frac{\zeta}{\left(x^2+\frac{\zeta}{2}\right)(x^2+\zeta)} \bar{z}_\alpha dz_\alpha$ and $\partial_z = dz_\alpha \frac{\partial}{\partial z_\alpha}$, $\bar{\partial}_{\bar{z}} = d\bar{z}_\alpha \frac{\partial}{\partial \bar{z}_\alpha}$ $(d = \partial_z + \bar{\partial}_{\bar{z}})$. Then the corresponding field strength F_A is obtained from

$$\begin{aligned} F_A &= dA + A^2, \\ &= \xi^{-1}(d\alpha + \alpha^2)\xi. \end{aligned} \tag{21}$$

Even if we deal with the $U(1)$ case we have to keep the second term in (21) because of the gauge covariance on noncommutative space. The field strength F_A can be obtained from (14) or by direct calculation of (21) with attention to ordering:

$$\begin{aligned} &F_A = \frac{\zeta}{x^2\left(x^2+\frac{\zeta}{2}\right)(x^2+\zeta)} \Big(f_3(dz_2 d\bar{z}_2 - dz_1 d\bar{z}_1) + f_+ d\bar{z}_1 dz_2 + f_- d\bar{z}_2 dz_1 \Big), \\ &f_3 = z_1\bar{z}_1 - z_2\bar{z}_2, \quad f_+ = 2z_1\bar{z}_2, \quad f_- = 2z_2\bar{z}_1. \end{aligned} \tag{22}$$

Since the "origin", i.e. $|0,0\rangle$, is projected out, F_A has no singularity. The topological action density is given by

$$\hat{S} = -\frac{1}{8\pi^2} F_A F_A = -\frac{\zeta^2}{\pi^2} \frac{1}{x^2\left(x^2+\frac{\zeta}{2}\right)^2 (x^2+\zeta)} p, \tag{23}$$

where we used the fact that $dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = -4$ (volume form). Note that $\hat{S}$ for $k=1$ depends only on x^2 due to the rotational symmetry.

The total action over noncommutative $\mathbf{R}^4$ is defined by using the prescription (15) as

$$\mathrm{Tr}_{\mathcal{H}} \hat{S} = \left(\frac{\zeta\pi}{2}\right)^2 \sum_{(n_1,n_2)\neq(0,0)}^{\infty} \hat{S}. \tag{24}$$

Using the facts

$$\begin{aligned} x^2|n_1,n_2\rangle &= \frac{\zeta}{2}(n_1+n_2)|n_1,n_2\rangle, \\ \sum_{(n_1,n_2)\neq(0,0)}^{\infty} \langle N|\mathcal{O}(x)|N\rangle &= \sum_{N=1}^{\infty} (N+1)\langle N|\mathcal{O}(x)|N\rangle, \end{aligned} \tag{25}$$

the instanton number for the solution (20) turns out to be -1, that is,

$$\mathrm{Tr}_{\mathcal{H}} \hat{S} = -4 \sum_{N=1}^{\infty} \frac{1}{N(N+1)(N+2)} = -1. \tag{26}$$

3.2 Two $U(1)$ Instantons

Next we will perform the same calculation for $U(1)$ instanton solution with $k=2$. We start with the matrices satisfying the ADHM constraints (5) and (6):

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} \\ 0 & 0 \end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix} 0 \\ \sqrt{2\zeta} \end{pmatrix}, \quad J = 0, \tag{27}$$

where we have fixed the moduli corresponding to the relative position between two instantons for simplicity (for a general solution with the moduli dependence, see [11]). With these data we can get the normalized solution of Eq.(11)

$$
\begin{aligned}
&\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix}, \\
&\psi_1 = \sqrt{2\zeta} \begin{pmatrix} \sqrt{\zeta}\bar{z}_1\bar{z}_2 \\ \bar{z}_2(x^2+\frac{\zeta}{2}) \end{pmatrix} Q^{-1}, \quad \psi_2 = \sqrt{2\zeta} \begin{pmatrix} \sqrt{\zeta}\bar{z}_1\bar{z}_1 \\ \bar{z}_1(x^2-\frac{\zeta}{2}) \end{pmatrix} Q^{-1}, \\
&\xi = \left[\frac{-\zeta z_1\bar{z}_1 + x^2(x^2+\frac{1}{2}\zeta)}{-\zeta z_1\bar{z}_1 + (x^2+\frac{\zeta}{2})(x^2+2\zeta)} \right]^{\frac{1}{2}},
\end{aligned}
\tag{28}
$$

where

$$
Q = \left[\{-\zeta z_1\bar{z}_1 + x^2(x^2+\frac{1}{2}\zeta)\}\{-\zeta z_1\bar{z}_1 + (x^2+\frac{\zeta}{2})(x^2+2\zeta)\} \right]^{\frac{1}{2}}.
$$

Note that the states $|0,0\rangle$ and $|1,0\rangle$ are annihilated by all components of ψ [11] (where the projected states in general depend on the moduli entering in B_1 and B_2). The operator ψ in (28) is thus normalized as

$$
\psi^\dagger\psi = p \equiv 1 - |0,0\rangle\langle 0,0| - |1,0\rangle\langle 1,0|, \tag{29}
$$

so Q^{-1} is well-defined.

With this solution, the field strength F_A can be calculated with careful ordering from the formula (14)

$$
\begin{aligned}
&F_A = \frac{2\zeta}{Q}\Big(f_3(dz_2 d\bar{z}_2 - dz_1 d\bar{z}_1) + f_+ d\bar{z}_1 dz_2 + f_- d\bar{z}_2 dz_1 \Big), \\
&f_3 = \frac{G_1}{QP_1} z_1\bar{z}_1 - \frac{G_2}{QP_2} z_2\bar{z}_2, \quad f_+ = \frac{2(x^2+\frac{\zeta}{2})H_1}{Q_1P_1} z_1\bar{z}_2, \quad f_- = \frac{2(x^2+\frac{\zeta}{2})H_2}{Q_2P_2} z_2\bar{z}_1, \\
&G_1 := \zeta(3x^2+\frac{\zeta}{2})(z_1\bar{z}_1-\frac{\zeta}{2}) + x^2(x^2-\frac{\zeta}{2})^2, \\
&G_2 := \zeta(3x^2+\frac{5}{2}\zeta)z_1\bar{z}_1 + x^2(x^2+\frac{\zeta}{2})^2, \\
&H_1 := 3\zeta(z_1\bar{z}_1-\frac{\zeta}{2}) + x^2(x^2-\frac{\zeta}{2}), \\
&H_2 := 3\zeta z_1\bar{z}_1 + x^2(x^2-\frac{\zeta}{2}), \\
&Q_1 := \left[\{-\zeta(z_1\bar{z}_1-\frac{\zeta}{2}) + x^2(x^2+\frac{\zeta}{2})\}\{-\zeta(z_1\bar{z}_1-\frac{\zeta}{2}) + (x^2+\frac{\zeta}{2})(x^2+2\zeta)\} \right]^{\frac{1}{2}}, \\
&Q_2 := \left[\{-\zeta(z_1\bar{z}_1+\frac{\zeta}{2}) + x^2(x^2+\frac{\zeta}{2})\}\{-\zeta(z_1\bar{z}_1+\frac{\zeta}{2}) + (x^2+\frac{\zeta}{2})(x^2+2\zeta)\} \right]^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
&P_1 := -\zeta(z_1\bar{z}_1 - \frac{\zeta}{2}) + x^2(x^2 + \frac{3}{2}\zeta), \\
&P_2 := -\zeta z_1\bar{z}_1 + x^2(x^2 + \frac{3}{2}\zeta).
\end{aligned} \tag{30}
$$

The f_- component has a singularity coming from Q_2 at $|0,1\rangle$ state which is not projected out by p. However, F_A is well-defined since the $|0,1\rangle$ state is annihilated before it causes any trouble due to the factor $z_2\bar{z}_1$ in f_-. Notice that, in the case of $k=2$ instanton solution with relative separation, we can not expect the spherical symmetry, so the action depends on another coordinates in addition to x^2.

By straightforward calculation, the instanton charge density $\hat{S}$ can be explicitly calculated using the same normalization that the $k=1$ case

$$
\begin{aligned}
\hat{S} \;&=\; -\frac{1}{8\pi^2}F_AF_A \qquad (31) \\
&=\; -\frac{4\zeta^2}{\pi^2}\frac{1}{Q^2}\Big[\frac{1}{Q^2}\Big(\frac{G_1}{P_1}z_1\bar{z}_1 - \frac{G_2}{P_2}z_2\bar{z}_2\Big)^2 + \frac{2(x^2+\frac{\zeta}{2})^2H_1^2}{Q_1^2P_1^2}z_1\bar{z}_1\bar{z}_2z_2 + \frac{2(x^2+\frac{\zeta}{2})^2H_2^2}{Q_2^2P_2^2}\bar{z}_1z_1z_2\bar{z}_2\Big]p.
\end{aligned}
$$

Now the instanton charge can be numerically calculated in the SHO basis (4) (where the sum with respect to n_1 and n_2 should be separately done since the spherical symmetry is broken). We performed this double infinite sum using Maple over $40,399$ points with $0 \le n_1 \le 200,\ 0 \le n_2 \le 200$ excluding the indicated points $(0,0),\ (1,0)$. The result is

$$
\mathrm{Tr}_{\mathcal{H}}\hat{S} = \Big(\frac{\zeta\pi}{2}\Big)^2 \sum_{\substack{(n_1,n_2)\neq(0,0)\\(n_1,n_2)\neq(1,0)}}^{\infty} \hat{S} \approx -1.9998877 \approx -2. \tag{32}
$$

Now this result is consistent with [14]. [2] Following the argument in [14], we believe that the instanton number for $U(1)$ solutions is always an integer, independent of the moduli entering in B_1, B_2. This should be the case since we have already introduced the integer number k to specify the ADHM data.

3.3 Single $U(2)$ Instanton

Now we will seek for $U(2)$ solution [11] following the same steps as the $U(1)$ case. From the ADHM equations with $B_1 = B_2 = 0$, one can choose I and J as follows

$$
\begin{aligned}
&I = \begin{pmatrix}\sqrt{\rho^2+\zeta} & 0\end{pmatrix} := \begin{pmatrix} a & 0\end{pmatrix}, \\
&J = \begin{pmatrix}0\\ \rho\end{pmatrix} := \begin{pmatrix}0\\ b\end{pmatrix},
\end{aligned} \tag{33}
$$

[2] In the previous version of this paper, we obtained $\mathrm{Tr}_{\mathcal{H}}\hat{S} = -0.932$ incorrectly due to an error of our numerical calculation.

where ρ is a non-negative number and parameterizes the classical size of the instanton. Then, from Eq.(11), we get the following solution

$$\begin{aligned}
\psi_1 &= \bar{z}_2\delta^{-1}I\xi - z_1\Delta^{-1}J^\dagger\xi, \\
\psi_2 &= \bar{z}_1\delta^{-1}I\xi + z_2\Delta^{-1}J^\dagger\xi, \\
\xi &= (1 + I^\dagger\delta^{-1}I + J\Delta^{-1}J^\dagger)^{-\frac{1}{2}},
\end{aligned} \tag{34}$$

where $\Delta = \delta + \zeta$. Using the explicit solution (33), ξ is expressed as

$$\xi = \begin{pmatrix} \left(\frac{\delta}{\Delta+\rho^2}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{\Delta}{\Delta+\rho^2}\right)^{\frac{1}{2}} \end{pmatrix} := \begin{pmatrix} \xi_- & 0 \\ 0 & \xi_+ \end{pmatrix}$$

and the zero-modes ψ are

$$\begin{aligned}
\psi_1 &:= \begin{pmatrix} f_1 & g_1 \end{pmatrix} = \begin{pmatrix} \bar{z}_2(\frac{1}{\delta}a\xi_-) & -z_1(\frac{1}{\Delta}b\xi_+) \end{pmatrix}, \\
\psi_2 &:= \begin{pmatrix} f_2 & g_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1(\frac{1}{\delta}a\xi_-) & z_2(\frac{1}{\Delta}b\xi_+) \end{pmatrix}, \\
\psi &:= \begin{pmatrix} \psi^{(1)} & \psi^{(2)} \end{pmatrix} = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \\ \xi_- & 0 \\ 0 & \xi_+ \end{pmatrix}.
\end{aligned} \tag{35}$$

From the above expression we see that $\psi^{(1)}$ annihilates the state $|0,0\rangle$ for any values of ρ, and normalized in the subspace where $|0,0\rangle$ is projected out, that is, $\psi^{(1)\dagger}\psi^{(1)} = p$. The zero-mode $\psi^{(2)}$ annihilates no states in $\mathcal{H}$ and manifestly nonsingular even if $\rho = 0$. When $\rho = 0$, $g_1 = g_2 = 0$, and, from (14), we see that $\psi^{(2)}$ does not contribute to the field strength. Therefore the structure of the $U(2)$ instanton at $\rho = 0$ is completely determined by the minimal zero-mode $\psi^{(1)}$ in the $U(1)$ subgroup [11].

The gauge field $A = \psi^\dagger d\psi$ can be now explicitly calculated and the result is

$$\begin{aligned}
A &= \xi^{-1}\alpha\xi + \xi^{-1}d\xi, \\
\alpha &= K\begin{pmatrix} C_1\bar{z}_\alpha dz_\alpha & C_2(z_1dz_2 - z_2dz_1) \\ C_2(\bar{z}_2d\bar{z}_1 - \bar{z}_1d\bar{z}_2) & C_3z_\alpha d\bar{z}_\alpha \end{pmatrix},
\end{aligned} \tag{36}$$

where

$$K = \frac{1}{(\delta + \frac{\zeta}{2})(\Delta + \rho^2)}, \quad C_1 = -(\rho^2 + \zeta), \quad C_2 = \rho\sqrt{\rho^2 + \zeta}, \quad C_3 = -\rho^2.$$

If we let $\zeta = 0$, we can get the ordinary $SU(2)$ instanton solution

$$A_\mu = -2i\rho^2\Sigma_{\mu\nu}\frac{x_\nu}{x^2(x^2 + \rho^2)}, \tag{37}$$

where $\Sigma_{\mu\nu}$ is the 't Hooft symbol which is both antisymmetric and self-dual with respect to their indices [15]. On the other hand, if we let $\rho = 0$, we get

$$\alpha = -\frac{\zeta}{\left(x^2 + \frac{\zeta}{2}\right)(x^2 + \zeta)}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2), \tag{38}$$

which is exactly equal to the $U(1)$ solution in (20) for the reason explained above.

The field strength F_A can be obtained from (14) or by direct calculation with the solution (36) if one keeps in mind careful ordering [3]:

$$\begin{aligned} F_A = \quad & d\bar{z}_1 \wedge dz_1 \begin{pmatrix} \frac{1}{2}B_1(2\bar{2} - 1\bar{1}) & B_2(12) \\ B_3(\bar{1}\bar{2}) & \frac{1}{2}B_4(1\bar{1} - 2\bar{2}) \end{pmatrix} \\ + \quad & d\bar{z}_2 \wedge dz_2 \begin{pmatrix} \frac{1}{2}B_1(1\bar{1} - 2\bar{2}) & -B_2(12) \\ -B_3(\bar{1}\bar{2}) & \frac{1}{2}B_4(2\bar{2} - 1\bar{1}) \end{pmatrix} \\ + \quad & d\bar{z}_1 \wedge dz_2 \begin{pmatrix} -B_1(1\bar{2}) & -B_2(11) \\ B_3(\bar{2}\bar{2}) & B_4(1\bar{2}) \end{pmatrix} \\ + \quad & dz_1 \wedge d\bar{z}_2 \begin{pmatrix} B_1(\bar{1}2) & -B_2(22) \\ B_3(\bar{1}\bar{1}) & -B_4(\bar{1}2) \end{pmatrix}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} B_1 &:= \frac{2C_1}{\delta(\delta + \rho^2 + \frac{\zeta}{2})(\Delta + \rho^2)} \\ B_2 &:= \frac{2C_2}{\delta(\delta + \rho^2 + \frac{\zeta}{2})(\Delta + \rho^2)} \left(\frac{\Delta + \rho^2}{\delta + \rho^2}\right)^{\frac{1}{2}} \\ B_3 &:= \frac{2C_2}{\Delta(\Delta + \rho^2)(\Delta + \rho^2 + \frac{\zeta}{2})} \left(\frac{\Delta + \rho^2}{\Delta + \rho^2 + \zeta}\right)^{\frac{1}{2}} \\ B_4 &:= \frac{2C_3}{\Delta(\Delta + \rho^2)(\Delta + \rho^2 + \frac{\zeta}{2})}. \end{aligned}$$

One can check that this F_A is anti-Hermitian and anti-self-dual ($F^A_{1\bar{1}} + F^A_{2\bar{2}} = 0$) using the rule (19). By straightforward calculation, one can determine the instanton charge density which also depends only on x^2 due to rotational symmetry

$$\begin{aligned} \hat{S} \quad &= -\frac{1}{8\pi^2} Tr(F_A \wedge F_A) \\ &= -\frac{1}{2\pi^2}\Big\{(B_1^2 + B_4^2)(1\bar{1}\bar{2}2 + \bar{1}12\bar{2}) + B_2^2(11\bar{1}\bar{1} + 22\bar{2}\bar{2}) + B_3^2(\underbrace{\bar{1}\bar{1}11 + \bar{2}\bar{2}22})\Big\} \\ &\quad -\frac{1}{4\pi^2}\Big\{(B_1^2 + B_4^2)(1\bar{1} - 2\bar{2})^2 + 4B_2^2 1\bar{1}2\bar{2} + 4B_3^2\, \underbrace{\bar{1}1\bar{2}2}\Big\} \\ &= -\frac{1}{4\pi^2}\Big\{(B_1^2 + B_4^2)x^2\left(x^2 + \zeta\right) + 2B_2^2\Big(x^2 - \frac{\zeta}{2}\Big)x^2\Big\}p \\ &\quad -\frac{1}{2\pi^2}B_3^2\left(x^2 + \zeta\right)\left(x^2 + \frac{3\zeta}{2}\right). \end{aligned} \tag{40}$$

[3] Here we are using a shorthand notation where $(\alpha\alpha)$ denotes the coordinates $z_\alpha z_\alpha$ and $(\bar{\alpha}\alpha) = \bar{z}_\alpha z_\alpha$, etc.

In the above expression, the parts except $\underbrace{(\cdots)}$ project out the state $|0,0\rangle$, so we explicitly inserted the projection operator $p = 1 - |0,0\rangle\langle 0,0|$ in the parts. It can be confirmed again to recover the ordinary $SU(2)$ instanton solution in the $\zeta = 0$ limit where $B_1 = \cdots = B_4$ and the $U(1)$ case for the limit $\rho = 0$ where only B_1 term in (40) survives.

Finally we calculate the instanton charge of $U(2)$ solution. Note that, since the part involved with B_3^2 in (40), denoted as $\hat{S}_2$, does not project out any states in $\mathcal{H}$, the trace with respect to the part should be performed over the full Hilbert space (4) including the state $|0,0\rangle$, while that involved with the projection operator p in (40), denoted as $\hat{S}_1$, has to exclude the "origin", $|0,0\rangle$. Using the relations (25), the topological charge can be calculated (where we used Maple)

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}\hat{S} &= \left(\frac{\zeta\pi}{2}\right)^2 \left(\sum_{N=1}^{\infty}(N+1)\hat{S}_1(N) + \sum_{N=0}^{\infty}(N+1)\hat{S}_2(N)\right) \qquad (41) \\ &= \sum_{N=1}^{\infty}\Bigg(\frac{4}{(N+1)(N+2)(N+2a^2)(N+2a^2+1)^2(N+2a^2+2)^2(N+2a^2+3)^2} \times \\ &\quad \Big((N^3+6N^2+11N+6)^2 + 2a^2(N+2)^3(3N^3+16N^2+25N+12) + \\ &\quad 2a^4(3N^6+45N^5+257N^4+714N^3+1028N^2+737N+212) + \\ &\quad 4a^6(9N^5+83N^4+301N^3+512N^2+400N+115) + \\ &\quad 8a^8(9N^4+55N^3+122N^2+109N+30) + 16a^{10}(3N^3+11N^2+12N+3)\Big)\Bigg) \\ &= -1, \end{aligned}$$

where $a = \rho/\sqrt{\zeta}$. Note that the dependence on the instanton modulus ρ remarkably dissappears in the final answer. [4]

4 Completeness Relation

In this section we will show that our solutions we constructed in section 3 exactly satisfy the completeness relation (12). And then it is shown that this completeness relation is a general property satisfied in the ADHM construction.

The completeness relation (12) is actually a canonical decomposition of a vector space $\mathbf{C}^{N+2k}$ (or a free module $\mathcal{A}^{\otimes N+2k}$ for noncommutative instantons) into the null-space (11) and its orthogonal complement. This decomposition is well-defined [12] even in the noncommutative space in spite of the nontrivial normalization (13) since the projective module (see footnote 1) corresponding to a vector bundle is also well-defined in this case.

[4] Now this result is in agreement with the result in [14]. In the previous version of this paper, it was incorrectly claimed due to a programming error that the instanton number depends on the moduli.

Let's start with the simplest case, the single $U(1)$ instanton in section 3.1. In this case, the matrices $DfD^\dagger$ and $\psi\psi^\dagger$ where $f^{-1} = z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta = \Delta$ have the forms

$$DfD^\dagger = \begin{pmatrix} z_1 f\bar{z}_1 + \bar{z}_2 f z_2 & \bar{z}_2 f z_1 - z_1 f \bar{z}_2 & -\sqrt{\zeta}\bar{z}_2 f \\ \bar{z}_1 f z_2 - z_2 f\bar{z}_1 & \bar{z}_1 f z_1 + z_2 f\bar{z}_2 & -\sqrt{\zeta}\bar{z}_1 f \\ -\sqrt{\zeta} f z_2 & -\sqrt{\zeta} f z_1 & \zeta f \end{pmatrix} \tag{42}$$

$$\psi\psi^\dagger = \begin{pmatrix} \bar{z}_2\frac{\zeta}{\delta\Delta}z_2 & \bar{z}_2\frac{\zeta}{\delta\Delta}z_1 & \bar{z}_2\frac{\sqrt{\zeta}}{\Delta} \\ \bar{z}_1\frac{\zeta}{\delta\Delta}z_2 & \bar{z}_1\frac{\zeta}{\delta\Delta}z_1 & \bar{z}_1\frac{\sqrt{\zeta}}{\Delta} \\ \frac{\sqrt{\zeta}}{\Delta}z_2 & \frac{\sqrt{\zeta}}{\Delta}z_1 & \frac{\delta}{\Delta} \end{pmatrix} \tag{43}$$

To check the completeness relation (12) is now a simple straightforward algebra using the formula (19). Similarly one can easily check the completeness relation for the $U(2)$ single instanton in section 3.3. For this case,

$$DfD^\dagger = \begin{pmatrix} z_1 f\bar{z}_1 + \bar{z}_2 f z_2 & \bar{z}_2 f z_1 - z_1 f\bar{z}_2 & -\sqrt{\rho^2+\zeta}\bar{z}_2 f & \rho z_1 f \\ \bar{z}_1 f z_2 - z_2 f\bar{z}_1 & \bar{z}_1 f z_1 + z_2 f\bar{z}_2 & -\sqrt{\rho^2+\zeta}\bar{z}_1 f & -\rho z_2 f \\ -\sqrt{\rho^2+\zeta} f z_2 & -\sqrt{\rho^2+\zeta} f z_1 & (\rho^2+\zeta) f & 0 \\ \rho f\bar{z}_1 & -\rho f\bar{z}_2 & 0 & \rho^2 f \end{pmatrix} \tag{44}$$

$$\psi\psi^\dagger = \begin{pmatrix} f_1 f_1^\dagger + g_1 g_1^\dagger & f_1 f_2^\dagger + g_1 g_2^\dagger & f_1\xi_-^\dagger & g_1\xi_+^\dagger \\ f_2 f_1^\dagger + g_2 g_1^\dagger & f_2 f_2^\dagger + g_2 g_2^\dagger & f_2\xi_-^\dagger & g_1\xi_+^\dagger \\ \xi_- f_1^\dagger & \xi_- f_2^\dagger & \xi_-\xi_-^\dagger & 0 \\ \xi_+ g_1^\dagger & \xi_+ g_2^\dagger & 0 & \xi_+\xi_+^\dagger \end{pmatrix} \tag{45}$$

where $f^{-1} = z_1\bar{z}_1 + z_2\bar{z}_2 + \rho^2 + \zeta = \Delta + \rho^2$ and the notations in (45) are coming from (35).

Before checking the completeness for the case of the $U(1)$ two instantons, let's argue that the completeness relation (12) in the ADHM construction is a general property satisfied even for noncommutative spaces. For this, it is important to observe the following property

$$\psi p = \psi, \quad p\psi^\dagger = \psi^\dagger, \tag{46}$$

where p is the projection operator in (13) which is $p = 1 - |0,0\rangle\langle 0,0|$ for $k=1$ $U(1)$ instanton and $p = \begin{pmatrix} 1-|0,0\rangle\langle 0,0| & 0 \\ 0 & 1 \end{pmatrix}$ for $k=1$ $U(2)$ instanton. Of course the above property is quite general in the ADHM construction for the reason stated below (19). When the null-space condition (11) is given in a vector space $\mathbf{C}^{N+2k}$ or a free module $\mathcal{A}^{\otimes N+2k}$, one can ask whether the completeness relation (12) is satisfied. If one notices the projection operators in left-hand side of (12) are both well-defined, i.e. non-singular, this relation should be satisfied since operating D and ψ from the right-hand side or $D^\dagger$ and $\psi^\dagger$ from the left-hand side the relation is always satisfied due to (11) and (46) respectively. One can check using the explicit expressions (42)-(45) that the relation (12) is satisfied, as it should be, even for the dangerous state $|0,0\rangle$. This general argument actually can also be extracted from the construction of a projective instanton module given by Ho [12]. We checked this claim for the $U(1)$ two instantons in section 3.2 as well although a little but straightforward algebra has been involved.

5 Discussion

We studied $U(1)$ and $U(2)$ instanton solutions on noncommutative $\mathbf{R^4}$ based on the noncommutative version of ADHM equation proposed by Nekrasov and Schwarz. It has been shown that the anti-self-dual gauge fields on self-dual noncommutative $\mathbf{R}^4$ correctly give integer instanton numbers for all cases we consider.

We further showed that the completeness relation in the ADHM construction is a general property satisfied even for noncommutative spaces. To illustrate this claim more concretely, let's consider the ADHM construction on $\mathbf{R}^2_{NC} \times \mathbf{R}^2_C$ where $\mathbf{R}^2_{NC}$ is the noncommutative space but $\mathbf{R}^2_C$ is the commutative space. This space is represented by the algebra

$$[\bar{z}_1, z_1] = \zeta, \quad [\bar{z}_2, z_2] = 0. \tag{47}$$

With this convention, one can easily check using the explicit expressions (42)-(45) that the completeness relation is exactly satisfied for this space. (Note that now $z_2,\ \bar{z}_2$ are commutative coordinates, so one should apply the formula (19) only for $\alpha = 1$ with the change $\zeta \to 2\zeta$.) Our result is different from [14] by Chu, et al. claiming that the completeness relation can be broken down in this space. Indeed they argued that there is no nonsingular $U(N)$ instanton on $\mathbf{R}^2_{NC} \times \mathbf{R}^2_C$ due to the breakdown of the completeness relation. Our present result may cure their "unexpected" result correctly. We hope to address this problem soon [16].

We would like to mention some difference on finding an ADHM solution between ours and [14] although both methods consequently give equivalent results. We have taken the normalization (13) throughout this paper and found the solutions (17), (28) and (35) satisfying this normalization. Alternatively one can take an another nomalization $\psi^\dagger \psi = 1$ like as [14]. In this case one should find the ADHM solution satisfying this normalization. Explicit solutions of this kind were found in [14]. However, the choice of normalization is not important if and only if the consistent solution could be found according to each normalization. In addition, as shown in section 4, our normalization (13) is completely consistent with the completeness relation due to the property (46) (and more economical on calculational side).

Acknowledgments

We would like to thank Pei-Ming Ho for useful discussions and comments and Koji Hashimoto, Eunsang Kim, and Satoshi Watamura for helpful discussions on related matter. We thank C.-S. Chu, V. V. Khoze and G. Travaglini for their critical comments in [14]. We were supported by the Ministry of Education, BK21 Project No. D-0055 and by grant No. 1999-2-112-001-5 from the Interdisciplinary Research Program of the KOSEF. H.S.Y. was also supported in part by Basic Science Research Institute in Sogang University.

References

[1] A. Connes, M. R. Douglas and A. Schwarz, J. High Energy Phys. **02** (1998) 003.

[2] M. R. Douglas and C. Hull, J. High Energy Phys. **02** (1998) 008.

[3] N. Seiberg and E. Witten, J. High Energy Phys. **09** (1999) 032.

[4] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112.

[5] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. **B498** (1997) 467.

[6] N. A. Nekrasov and A. Schwarz, Comm. Math. Phys. **198** (1998) 689.

[7] M. Atiyah, N. Hitchin, V. Drinfeld and Y. Manin, Phys. Lett. **65A** (1978) 185; E. Corrigan and P. Goddard, Ann. Phys. (N.Y.) **154** (1984) 253; S. Donaldson, Comm. Math. Phys. **93** (1984) 453.

[8] M. Berkooz, Phys. Lett. **B430** (1998) 237.

[9] H. Nakajima, Ann. of Math. **145** (1997) 379; Nucl. Phys. Proc. Suppl. **46** (1996) 154.

[10] H. W. Braden and N. A. Nekrasov, Spacetime Foam from Noncommutative Instantons, `hep-th/9912019`.

[11] K. Furuuchi, Prog. Theor. Phys. **103** (2000) 1043.

[12] P.-M. Ho, Twisted Bundle on Noncommuative Space and $U(1)$ Instanton, `hep-th/0003012`.

[13] D. H. Correa, G. S. Lozano, E. F. Moreno and F. A. Schaposnik, Comments on the U(2) Noncommutative Instanton, `hep-th/0105085`.

[14] C.-S. Chu, V. V. Khoze and G. Travaglini, Notes on Noncommutative Instantons, `hep-th/0108007`.

[15] R. Rajaraman, *Solitons and Instantons* (North-Holland Publishing, 1982).

[16] K.-Y. Kim, B.-H. Lee and H. S. Yang, in preparation.